Multiplicative Torsion and Axial Noether Charge

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Some times ago, a Lagrangian density has been proposed by the author where only the local symmetries of the Lorentz subgroup of (A)ds group is retained. This formalism has been found to produce some results encompassing that of standard Einstein-Hilbert formalism. In the present article, the conserved axial vector matter currents, constructed in some earlier paper, have been found to be a result of Noether's theorem.

PACS numbers:
Keywords:

1. INTRODUCTION

It is well known[1] that torsion and curvature of any manifold are related to translation and rotation respectively. In particular torsion is more precisely related to broken translation gauge fields within the framework of nonlinear realization of the local space time group[2]. So to exploit both the symmetries of translation and rotation any gravitational Lagrangian must contain torsion together with curvature.

Kibble [3] and Sciama[4] pointed out that the *Poincaré* group, which is the semi-direct product of translation and Lorentz rotation, is the underlying gauge group of gravity and found the so-called Einstein-Cartan theory where mass-energy of matter is related to the curvature and spin of matter is related to the torsion of space-time. From geometrical point of view there is an important connection between the de Sitter group and the *Poincaré* group. It is a well known fact that the *Poincaré* group can be obtained from the de Sitter group by an appropriate *Inönü-Wigner* contraction[5, 6]. In the late 1970s, MacDowell and Mansouri[7] introduced a new approach of gravity, based on broken symmetry in a type of gauge theory. Macdowell-Mansouri gravity is based on the (A)dS group which retains only the local symmetries of the Lorentz subgroup. Even at the level of the action, the exact local (A)dS symmetry is clearly broken, whereas local Lorentz symmetry is retained.

In differential geometry we know that certain global features of a manifold are determined by some local invariant densities. These topological invariants have an important property in common - they are total divergences and in any local theory these invariants, when treated as Lagrangian densities, contribute nothing to the Euler-Lagrange equations. Hence in a local theory only few parts, not the whole part, of these invariants may be kept in a Lagrangian density. Few years ago, a gravitational Lagrangian has been proposed[8] where a Lorentz invariant part of the de Sitter Pontryagin density has been treated as the Einstein-Hilbert Lagrangian. By this way the role of torsion in the underlying manifold has become multiplicative rather than additive one and the Lagrangian looks like **torsion** $\otimes$ **curvature**. In other words - the additive torsion is decoupled from the theory but not the multiplicative one. This indicates that torsion is uniformly nonzero everywhere. In the geometrical sense, this implies that microlocal space-time is such that at every point there is a direction vector (vortex line) attached to it. This effectively corresponds to the non commutative geometry having the manifold $M_4 \times Z_2$, where the discrete space Z_2 is just not the two point space[9], but appears as an attached direction vector. Considering torsion and torsion-less connection as independent fields[10], it has been found that κ of Einstein-Hilbert Lagrangian, appears as an integration constant in such a way that it has been found to be linked with the topological Nieh-Yan density of U_4 space. If we consider axial vector torsion together with a scalar field ϕ connected to a local scale factor[11], then the Euler-Lagrange equations not only give the constancy of the gravitational constant but they also link, in laboratory scale, the mass of the scalar field with the Nieh-Yan density and, in cosmic scale of Friedmann-Robertson-Walker(FRW)-cosmology, they predict only three kinds of the phenomenological energy density representing mass, radiation and cosmological constant. In a recent paper[12], it has been shown that this scalar field may also be interpreted to be linked with the dark matter and dark radiation. Also it has been shown that, using field equations of all fields except the frame field, the starting Lagrangian reduces to a generic $f(\mathcal{R})$ gravity Lagrangian which, for FRW metric, gives standard FRW cosmology[13]. Here $\mathcal{R}$ is the Ricci scalar of the Riemann curvature tensor. But for non-FRW metric, in particular of [14], with some particular choice of the functions of the scalar field ϕ one gets $f(\mathcal{R}) = f_0\mathcal{R}^{1+v_{tg}^2}$, where v_{tg} is the constant tangential velocity of the stars and gas clouds in circular orbits in the outskirts of spiral galaxies. With this choice of functions of ϕ no dark matter is required to explain flat galactic rotation curves. In a recent paper[15], we see that variation of torsion in the action gives us the axial vector 1-form $j_5 = \overline{\Psi}\gamma_5\gamma\Psi$ to be an exact form. If we consider FRW geometry to be in the background then the FRW postulate[15], makes it possible to define an axial vector current 3-form J_1^A as a product of torsion and matter current j_5. J_1^A is conserved and as well as gauge invariant. In manifolds having arbitrary background geometry the product of j_5 and an exact 2-form

F gives us another gauge invariant conserved current J_2^A. These conserved axial currents implies pseudoscalar conserved charges. In this article we are going to study the possible connection of the conserved current J_2^A with a Noether symmetry together with its physical significance in connection with the charge associated with the electromagnetic field and with the entropy of a black hole. In the following four sections we have studied the formalism employed in [11, 12, 15]. The section preceeding the section of discussion contains new results.

2. AXIAL VECTOR TORSION AND GRAVITY

Cartan's structural equations for a Riemann-Cartan space-time U_4 are given by[16, 17],

$$T^a = de^a + \omega^a{}_b \wedge e^b \quad (1)$$
$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \quad (2)$$

here $\omega^a{}_b$ and e^a represent the spin connection and the local frame respectively.

In U_4 there exists two invariant closed four forms. One is the well known Pontryagin[18, 19], density P and the other is the less known Nieh-Yan density N[20], given by

$$\begin{aligned} P &= R^{ab} \wedge R_{ab} && (3)\\ \text{and}\ N &= d(e_a \wedge T^a) \\ &= T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b. && (4) \end{aligned}$$

Here we consider a particular class of the Riemann-Cartan geometry where only the axial vector part of the torsion is nontrivial. Then from (4), using antisymmetric property of axial torsion, we can write the Nieh-Yan density as

$$\begin{aligned} N &= -R_{ab} \wedge e^a \wedge e^b = -{}^*N\eta\ , && (5)\\ \text{where}\ T^a \wedge T_a &= T^a{}_{bc} T_{adf} e^b \wedge e^c \wedge e^d \wedge e^f = 0 && (6)\\ \text{and}\ \eta &:= \frac{1}{4!}\epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d && (7) \end{aligned}$$

is the invariant volume element. It follows that ${}^*N$, the Hodge dual of N, is a scalar density of dimension $(length)^{-2}$.

We can combine the spin connection and the vierbeins multiplied by a scalar field together in a connection for $SO(4,1)$, in the tangent space, in the form

$$W^{AB} = \begin{bmatrix} \omega^{ab} & \phi e^a \\ -\phi e^b & 0 \end{bmatrix}, \quad (8)$$

where $a, b = 0, 1, ...3$; $A, B = 0, 1, 2, ...4$ and ϕ is a variable parameter of dimension $(length)^{-1}$ corresponding to a local length scale. In some earlier works, [8, 10, 21], ϕ has been treated as an inverse length constant. With this connection we can obtain $SO(4,1)$ Pontryagin density as[11]

$$\begin{aligned} F^{AB} \wedge F_{AB} &= R^{ab} \wedge R_{ab} + 2\phi^2 d(e^a \wedge T_a) \\ &\quad + 4\phi d\phi \wedge e^a \wedge T_a \\ &= P + dC_{T\phi}, && (9) \end{aligned}$$

where

$$\begin{aligned} C_{T\phi} &:= 2\phi^2 e^a \wedge T_a, && (10)\\ P &:= -R^a{}_b \wedge R^b{}_a \\ &= -(\bar{R}^a{}_b \wedge \bar{R}^b{}_a + 2\bar{R}^a{}_b \wedge \hat{R}^b{}_a \\ &\quad + \hat{R}^a{}_b \wedge \hat{R}^b{}_a), && (11)\\ \bar{R}^b{}_a &= d\bar{\omega}^b{}_a + \bar{\omega}^b{}_c \wedge \bar{\omega}^c{}_a, && (12)\\ \hat{R}^b{}_a &= dT^b{}_a + \bar{\omega}^b{}_c \wedge T^c{}_a \\ &\quad + T^b{}_c \wedge \bar{\omega}^c{}_a + T^b{}_c \wedge T^c{}_a && (13)\\ \text{and}\quad T^a{}_b &= \omega^a{}_b - \bar{\omega}^a{}_b \text{ s. t. } T^a{}_b \wedge e^b = T^a && (14) \end{aligned}$$

Now $-\bar{R}^a{}_b \wedge \bar{R}^b{}_a$, the purely Riemannian torsion-less part of P, is a closed four form and is given by

$$\begin{aligned} -\bar{R}^a{}_b \wedge \bar{R}^b{}_a &= -d(\bar{\omega}^a{}_b \wedge \bar{R}^b{}_a \\ &\quad -\frac{1}{3}\bar{\omega}^a{}_b \wedge \bar{\omega}^b{}_c \wedge \bar{\omega}^c{}_a) = dC_R && (15)\\ \text{where}\ C_R &= -(\bar{\omega}^a{}_b \wedge \bar{R}^b{}_a - \frac{1}{3}\bar{\omega}^a{}_b \wedge \bar{\omega}^b{}_c \wedge \bar{\omega}^c{}_a). \end{aligned}$$

With the hypothesis that only the axial vector part of the torsion is nontrivial, we can write

$$\begin{aligned} T^a &= e^{a\mu} T_{\mu\nu\alpha} dx^\nu \wedge dx^\alpha,\ \ T^{ab} = e^{a\mu} e^{b\nu} T_{\mu\nu\alpha} dx^\alpha \\ \text{and}\ {}^*A &= T = \frac{1}{3!} T_{\mu\nu\alpha} dx^\mu \wedge dx^\nu \wedge dx^\alpha && (16)\\ \text{s.t.}\quad & N = 6dT && (17) \end{aligned}$$

In this framework we see that

$$\begin{aligned} \hat{R}^a{}_b \wedge \hat{R}^b{}_a &= -2d(A \wedge dA \\ &\quad -\frac{1}{3} T^a{}_b \wedge T^b{}_c \wedge T^c{}_a) \\ &= -dC_T && (18)\\ \text{and}\quad 2\bar{R}^a{}_b \wedge \hat{R}^b{}_a &= -4\mathcal{R} dT + 8\mathcal{R}^{ab} \bar{\nabla}(A_b \eta_a) \\ &= 8d(G^{ab} A_b \eta_a) = -dC_{RT} && (19)\\ \text{where}\quad \eta_a &= \frac{1}{3!}\epsilon_{abcd} e^b \wedge e^c \wedge e^d, && (20)\\ C_T &= 2(A \wedge dA - \frac{1}{3} T^a{}_b \wedge T^b{}_c \wedge T^c{}_a) \\ \text{and}\ C_{RT} &= -8(G^{ab} A_b \eta_a). \end{aligned}$$

Here $\bar{\nabla}$ is the torsion-free covariant derivative; $\mathcal{R}$, $\mathcal{R}^{ab}$ and G^{ab} are, respectively, corresponding Ricci scalar, Ricci tensor and Einstein's tensor.

Hence we see that the $SO(4,1)$ Pontryagin density in U_4 is the sum of four closed four forms, given by

$$F^{AB} \wedge F_{AB} = dC_R + dC_T + dC_{RT} + dC_{T\phi}. \quad (21)$$

Since all these four forms are total divergences, they yield nothing in any local theory when treated as Lagrangian densities. Hence to have an effective field theory, however, we may consider some Lorentz invariant parts of them as Lagrangian densities. So here we heuristically

propose a Lagrangian density which combines a part of dC_{RT} with a part of $dC_{T\phi}$ as follows

$$\mathcal{L}_0 = (\mathcal{R} - \beta\phi^2)dT = -\frac{1}{6}(\mathcal{R} - \beta\phi^2)^* N\eta \tag{22}$$

where β is a dimensionless coupling constant.

So far $SO(3,1)$ invariance is concerned, torsion can be separated from the connection as the torsional part of the $SO(3,1)$ connection transforms like a tensor i.e. when vierbeins also transform like $SO(3,1)$ tensors in a broken $SO(4,1)$ gauge theory. In this direction it is important to define a torsion-free covariant differentiation through a field equation involving the connection and the vierbeins only and also we have to identify the torsion. So we introduce Lagrangian density $\mathcal{L}_1$ with two $SO(3,1)$ connections ω^{ab} and $\bar{\omega}^{ab}$, given by,

$$\begin{aligned}\mathcal{L}_1 \;=\; & {}^*[b_a \wedge (\overset{\omega}{\nabla} e^a - T^a)][b_a \wedge (\overset{\omega}{\nabla} e^a - T^a)] \\ & +{}^*[c_a \wedge (\omega^{ab} - \bar{\omega}^{ab} - T^{ab})][c_a \wedge \\ & (\omega^{ab} - \bar{\omega}^{ab} - T^{ab})],\end{aligned} \tag{23}$$

where $\overset{\omega}{\nabla}$ represents covariant differentiation with respect to the $SO(3,1)$ connection one form ω^{ab}, b_a and c_a are are respectively a 2-form and a 1-form with one internal index and of dimension $(length)^{-1}$. If we treat b_a and c_a as Lagrange multipliers then they define, respectively on the on-shell, T^a as the torsion 2-form and $\bar{\omega}^{ab}$ as the torsion free connection. By this way torsion becomes decoupled from the connection part of the theory. It becomes independent of the one form e^a and in particular, owing to its fundamental existence as a metric independent tensor in the affine connection in U_4, we treat here the three form $T = \frac{1}{3!}e^a \wedge T_a$ as more fundamental than the one form $T^{ab} = \omega^{ab} - \bar{\omega}^{ab}$.

Now we add another Lagrangian density $\mathcal{L}_2$ containing a nonlinear kinetic term for the scalar field ϕ, given by

$$\mathcal{L}_2 = -f(\phi)d\phi \wedge {}^*d\phi - h(\phi)\eta \tag{24}$$

where $f(\phi)$ and $h(\phi)$ are unknown functions of ϕ whose forms are to be determined subject to the geometric structure of the manifold.

Now we define the total gravitational Lagrangian density in empty space, as,

$$\begin{aligned}\mathcal{L}_G \;=\; & \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2, \\ =\; & -\frac{1}{6}({}^*N\mathcal{R}\eta + \beta\phi^2 N) + {}^*[b_a \wedge (\overset{\omega}{\nabla} e^a - T^a)][b_a \wedge \\ & (\overset{\omega}{\nabla} e^a - T^a)] + {}^*[c_a \wedge (\omega^{ab} - \bar{\omega}^{ab} - T^{ab})][c_a \\ & \wedge(\omega^{ab} - \bar{\omega}^{ab} - T^{ab})] \\ & -f(\phi)d\phi \wedge {}^*d\phi - h(\phi)\eta,\end{aligned} \tag{25}$$

where * is Hodge duality operator, $\mathcal{R}\eta = \frac{1}{2}\bar{R}^{ab} \wedge \eta_{ab}$, $\bar{R}^b{}_a = d\bar{\omega}^b{}_a + \bar{\omega}^b{}_c \wedge \bar{\omega}^c{}_a$, $T^a = e^{a\mu}T_{\mu\nu\alpha}dx^\nu \wedge dx^\alpha$, $T^{ab} = e^{a\mu}e^{b\nu}T_{\mu\nu\alpha}dx^\alpha$, $T = \frac{1}{3!}T_{\mu\nu\alpha}dx^\mu \wedge dx^\nu \wedge dx^\alpha$, $N = 6dT$, $\eta_a = \frac{1}{3!}\epsilon_{abcd}e^b \wedge e^c \wedge e^d$ and $\eta_{ab} = {}^*(e_a \wedge e_b)$.

Here β is a dimensionless coupling constant, $\bar{\nabla}$ represents covariant differentiation with respect to the connection one form $\bar{\omega}^{ab}$, b_a and c_a are are respectively a 2-form and a 1-form with one internal index and of dimension $(length)^{-1}$. $f(\phi)$, $h(\phi)$ are unknown functions of ϕ whose forms are to be determined subject to the geometric structure of the manifold. The geometrical implication of the first term, i.e. the $torsion \otimes curvature$ term, in the Lagrangian $\mathcal{L}_G$ has already been discussed in the beginning. Symmetry of the Lagrangian $\mathcal{L}_G$ is obviously "Lorentz Symmetry", since the gravitational Lagrangian, the first two terms of (25), is the sum of two Lorentz invariant parts of the de Sitter Pontryagin density. The Lagrangian $\mathcal{L}_G$ is only Lorentz invariant under rotation in the tangent space where de Sitter boosts are not permitted. Moreover, we define torsion only through the variation of the Lagrange multiplier b_a in the Lagrangian $\mathcal{L}_G$. As a consequence T can be treated independently of e^a and $\bar{\omega}^{ab}$. Here we note that, though torsion one form $T^{ab} = \omega^{ab} - \bar{\omega}^{ab}$ is a part of the $SO(3,1)$ connection, it does not transform like a connection form under $SO(3,1)$ rotation in the tangent space and thus it imparts no constraint on the gauge degree of freedom of the Lagrangian.

From the definition of the volume 4-form η we can write

$$\begin{aligned}\eta \;&=\; ed^4x, \\ &=\; \frac{1}{4!}e\epsilon_{\mu\nu\alpha\beta}dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta, \\ &=\; \frac{1}{4!}\tilde{\epsilon}_{\mu\nu\alpha\beta}, dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta, \text{ (say)},\end{aligned} \tag{26}$$

where $e = \det(e^a{}_\mu)$ is a pseudoscalar, i.e., it transforms as $\sqrt{-g}$, $g = \det(g_{\mu\nu})$, under continuous coordinate transformation but under coordinate reflection its sign changes accordingly. Any parity preserving action of gravity may be written as

$$\mathcal{A} = \int(e\mathcal{L}_S + \mathcal{L}_P)d^4x \tag{27}$$

Here $\mathcal{L}_S$ is a parity preserving scalar function of field variables and their derivatives where as $\mathcal{L}_P$ is a parity violating pseudo-scalar function. We have to note that, in the above action, the factor e is missing in front of $\mathcal{L}_P$. $\mathcal{L}_P$ itself changes its sign under odd parity transformation whereas the factor e is required for $\mathcal{L}_S$. Hence under arbitrary coordinate transformation $x \to \bar{x}$ we have

$$e\mathcal{L}_S d^4x \to \bar{e}\bar{\mathcal{L}}_S d^4\bar{x} \;=\; e\mathcal{L}_S d^4x \tag{28}$$

$$\text{and} \quad \mathcal{L}_P d^4x \to \bar{\mathcal{L}}_P d^4\bar{x} \;=\; \mathcal{L}_P d^4x \tag{29}$$

For example, we may consider the electromagnetic field $F_{\mu\nu}$ in curved space time. In this case $\mathcal{L}_S = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $\mathcal{L}_P = \frac{1}{8}F^{\mu\nu}F^{\alpha\beta}\epsilon_{\mu\nu\alpha\beta}$. In our analysis, we have to note that this argument holds only in curved space time but in Minkowski space there is no local frame field and therefore there is no e to absorb the sign change due to

reflection of coordinate axes. Hence in curved space time $e\mathcal{L}_S$ and $\mathcal{L}_P$ behave in the same manner under parity transformation but in Minkowskki space it is not true. Hence, in the present context where $\mathcal{R}$ and ${}^*N$ are scalar functions, the gravity Lagrangian $\mathcal{L}_G$ in (25) is invariant under parity.

3. SCALAR FIELD AND SPINORIAL MATTER

Total gravity Lagrangian in the presence of a spinorial matter field may be taken to be

$$\mathcal{L}_{tot.} = \mathcal{L}_G + \mathcal{L}_D, \tag{30}$$

where

$$\begin{aligned}\mathcal{L}_D &= \phi^2[\frac{i}{2}\{\overline{\psi}{}^*\gamma\wedge D\psi + \overline{D\psi}\wedge{}^*\gamma\psi\} - \frac{g}{4}\overline{\psi}\gamma_5\gamma\psi\wedge T \\ &\quad + c_\psi\sqrt{{}^*dT}\overline{\psi}\psi\eta] \qquad (31)\\ \gamma_\mu &:= \gamma_a e^a{}_\mu, \quad {}^*\gamma := \gamma^a\eta_a, \quad D := d+\Gamma \qquad (32)\\ \Gamma &:= \frac{1}{4}\gamma^\mu D^{\{\}}\gamma_\mu = \frac{1}{4}\gamma^\mu\gamma_{\mu:\nu}dx^\nu \\ &= -\frac{i}{4}\sigma_{ab}e^{a\mu}e^b{}_{\mu:\nu}dx^\nu \qquad (33)\end{aligned}$$

here $D^{\{\}}$, or : in tensorial notation, is Riemannian torsion free covariant differentiation acting on external indices only; $\sigma^{ab} = \frac{i}{2}(\gamma^a\gamma^b - \gamma^b\gamma^a)$, $\overline{\psi} = \psi^\dagger\gamma^0$ and g, c_ψ are both dimensionless coupling constants. Here ψ and $\overline{\psi}$ have dimension $(length)^{-\frac{1}{2}}$ and conformal weight $-\frac{1}{2}$. It can be verified that under $SL(2,C)$ transformation on the spinor field and gamma matrices, given by,

$$\begin{aligned}\psi\to\psi' &= S\psi, \quad \overline{\psi}\to\overline{\psi'} = \overline{\psi}S^{-1}\\ \text{and } \gamma\to\gamma' &= S\gamma S^{-1}, \qquad (34)\end{aligned}$$

where $S = \exp(\frac{i}{4}\theta_{ab}\sigma^{ab})$, Γ obeys the transformation property of a $SL(2,C)$ gauge connection, i.e.

$$\begin{aligned}\Gamma\to\Gamma' &= S(d+\Gamma)S^{-1} \qquad (35)\\ \text{s. t. } D\gamma &:= d\gamma + [\Gamma,\gamma] = 0. \qquad (36)\end{aligned}$$

Hence γ is a covariantly constant matrix valued one form w. r. t. the $SL(2,C)$ covariant derivative D. By Geroch's theorem, [22], we know that - the existence of the spinor structure is equivalent to the existence of a global field of orthonormal tetrads on the space and time orientable manifold. Hence use of Γ in the $SL(2,C)$ gauge covarint derivative is enough in a Lorentz invarint theory where de Sitter symmetry is broken.

In appendices A and B, by varying the independent fields in the Lagrangian $\mathcal{L}_{tot.}$, we obtain the Euler-Lagrange equations and then after some simplification we get the following results

$$\bar{\nabla}e_a = 0, \tag{A12$'$}$$
$${}^*N = \frac{6}{\kappa}, \tag{B10$'$}$$

i.e. $\bar{\nabla}$ is torsion free and κ is an integration constant having dimension of $(length)^2$.It is to be noted that, in (25), $\bar{\nabla}$ represents a $SO(3,1)$ covariant derivative, it is only on-shell torsion-free through the field equation (A12$'$). The $SL(2,C)$ covariant derivative represented by the operator D is torsion-free by definition, i.e. it is torsion-free both on on-shell and off-shell. Simultaneous and independent use of both $\bar{\nabla}$ and D in the Lagrangian density (30) has been found to be advantageous in the approach of this article. This amounts to the emergence of the gravitational constant κ to be only an on-shell constant and this justifies the need for the introduction of the Lagrangian multiplier b_a which appears twice in the Lagrangian density (25) such that $\bar{\omega}^a{}_b$ and e^a become independent fields.

$$m_\psi = c_\psi\sqrt{{}^*dT} = \frac{c_\psi}{\sqrt{\kappa}}, \tag{B12$'$}$$
$$\begin{aligned}&i{}^*\gamma\wedge D\Psi - \tfrac{g}{4}\gamma_5\gamma\wedge T\Psi + m_\Psi\Psi\eta = 0,\\ &i\overline{D\Psi}\wedge{}^*\gamma - \tfrac{g}{4}\overline{\Psi}\gamma_5\gamma\wedge T + m_\Psi\overline{\Psi}\eta = 0, \qquad (B13')\end{aligned}$$

where $\Psi = \phi\psi$ and $m_\Psi = m_\psi$.

$$\begin{aligned}G^b{}_a\eta &= -\kappa[\tfrac{i}{8}\{\overline{\Psi}(\gamma^b D_a + \gamma_a D^b)\Psi \\ &\quad -(\overline{D_a\Psi}\gamma^b + \overline{D^b\Psi}\gamma_a)\Psi\}\eta \\ &\quad -\tfrac{g}{16}\overline{\Psi}\gamma_5(\gamma_a{}^*T^b + \gamma^{b*}T_a)\Psi\eta \\ &\quad + f\partial_a\phi\partial^b\phi\eta + \tfrac{1}{2}(h)\eta\delta^b{}_a], \qquad (B19')\\ 0 &= [\tfrac{1}{2}\overline{\nabla}_\nu\overline{\Psi}\{\tfrac{\sigma^b{}_a}{2},\gamma^\nu\}\Psi \\ &\quad + \tfrac{i}{2}\{\overline{\Psi}(\gamma^b D_a - \gamma_a D^b)\Psi \\ &\quad -(\overline{D_a\Psi}\gamma^b - \overline{D^b\Psi}\gamma_a)\Psi\} \\ &\quad -\tfrac{g}{4}\overline{\Psi}\gamma_5(\gamma_a{}^*T^b - \gamma^{b*}T_a)\Psi]\eta, \qquad (B20')\end{aligned}$$

$$\begin{aligned}&\kappa d[\tfrac{g}{4}{}^*(\overline{\Psi}\gamma_5\gamma\Psi\wedge T) - f^*(d\phi\wedge{}^*d\phi) \\ &+2h - \tfrac{\beta}{\kappa}\phi^2] = -\tfrac{g}{4}\overline{\Psi}\gamma_5\gamma\Psi, \qquad (B21')\\ &\tfrac{2}{\kappa}\beta\phi\eta + f'(\phi)d\phi\wedge{}^*d\phi - h'(\phi)\eta + 2fd^*d\phi \\ &= -2\phi[\tfrac{i}{2}\{\overline{\psi}{}^*\gamma\wedge D\psi + \overline{D\psi}\wedge{}^*\gamma\psi\} \\ &\quad -\tfrac{g}{4}\overline{\psi}\gamma_5\gamma\psi\wedge T + m_\psi\overline{\psi}\psi\eta] = 0. \qquad (B22')\end{aligned}$$

Let us make few comments about these results,

- Right hand side of equation (B19$'$) may be interpreted, [23], as $(-\kappa)$ times the energy-momentum stress tensor of the Dirac field $\Psi(\overline{\Psi})$ together with the scalar field ϕ. Where by equation (B10$'$) the gravitational constant κ is $\frac{6}{{}^*N}$ and then by equation (B12$'$) mass of the spinor field is proportional to $\sqrt{{}^*N}$.

- Equation (B20$'$) represents covariant conservation of angular momentum of the Dirac field in the Einstein-Cartan space U_4 as a generalization of the same in the Minkwoski space M_4, [24].

- Equation (B22$'$) is the field equation of the scalar field ϕ. Here it appears that, in the on-shell, other than gravity, it has no source. Whereas in equation (B21$'$), there is a non trivial appearance of the torsion, the axial-vector matter-current and the scalar field ϕ; provided the coupling constant g is not negligible in a certain energy scale.

Since there is no significant experimental evidence of any torsion-matter interaction, [25], we may take, g to be negligible at present time although there is a possibility that g played a dominant role in the early universe. In other words, we may say that the scalar field, which appears to be connected with the spinor field only in equation (B21′), is at present playing the role of the dark matter and/or dark radiation. The consequence of this spin-torsion interaction term, in the very early universe, may be linked to the cosmological inflation without false vacuum, [26], primordial density fluctuation, [27, 28], and/or to the repulsive gravity, [31].

In teleparallel gravitational theories[29], torsion appears to be proportional to the spin density of matter. Absence of matter may imply no torsion. In this multiplicative-torsion approach 'torsion is uniformly nonzero everywhere', it may or may not couple with matter. It is a microscopic phenomenon with macroscopic significance. It's possible origin may be non-commutative geometry[9, 30]. It's everywhere presence makes it possible to define gravitational constant to be universal and as well as inversely proportional to the topological Nieh-Yan invariant. Again, it is to be remembered that present theory is a minimally extended theory of Einstein's GR. Role of other parts of torsion are not ruled out but they are not, simply, considered here. Hence in cosmological consideration, not in early Universe, torsion-matter coupling may be neglected without any prejudice and without affecting its (torsion's) multiplicative role. This prescription yields the phenomenological standard FRW-cosmology in the following section.

4. FRW-COSMOLOGY, SCALAR FIELD AND DARK MATTER

Here we analyse our results in the background of a FRW-cosmology where the metric tensor is given by

$$g_{00} = -1, \quad g_{ij} = \delta_{ij} a^2(t) \ \text{ where } \ i, j = 1, 2, 3; \tag{37}$$

such that

$$e = \sqrt{-\det(g_{\mu\nu})} = a^3 \tag{38}$$

Taking $g = 0$, equation (B21′) gives us

$$f\dot{\phi}^2 = -\frac{1}{\kappa}(\beta\phi^2 + \lambda) + 2h. \tag{39}$$

where λ is a constant of integration of dimension $(length)^{-2}$. Now, with the cosmological restriction on the metric as stated in (37) and the ϕ-field is a function of time only, the equation (B22′) reduces to

$$2f\ddot{\phi} + 2f\frac{e'}{e}\dot{\phi}^2 + f'\dot{\phi}^2 - \frac{2\beta}{\kappa}\phi + h' = 0 \tag{40}$$

where $'$ represents differentiation w. r. t. ϕ. If we eliminate $\ddot{\phi}$ from this equation with the help of the time derivative of equation (39), we get

$$\begin{aligned} 2f\frac{e'}{e}\dot{\phi}^2 &= \frac{4\beta}{\kappa}\phi - 3h' \\ \text{or,} \quad 2\frac{e'}{e} &= -\frac{\frac{4\beta}{\kappa}\phi - 3h'}{\frac{1}{\kappa}(\beta\phi^2 + \lambda) - 2h} \end{aligned} \tag{41}$$

For the FRW metric, the non-vanishing components of Einstein's tensor (B19′), w. r. t. external indices, are given by

$$\begin{aligned} G^0{}_0 &= -3(\frac{\dot{a}}{a})^2 = -\kappa(\rho_{BM} + \frac{\beta}{\kappa}\phi^2 + \frac{\lambda}{\kappa} - \frac{3h}{2}) \\ G^j{}_i &= -(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})\delta^j{}_i = -\kappa\frac{1}{2}(h)\delta^j{}_i \end{aligned} \tag{42}$$

where we have assumed that, in the cosmic scale, the observed (luminous) mass distribution is baryonic and co moving, s. t.

$$\begin{aligned} &\sum_{\Psi} \frac{i}{8}\{\overline{\Psi}(\gamma^b D_a + \gamma_a D^b)\Psi - (\overline{D_a\Psi}\gamma^b + \overline{D^b\Psi}\gamma_a)\Psi\} \\ &= \rho_{BM} = \frac{M_{BM}}{V}, \ \text{ for } a = b = 0, \\ &= 0, \ \text{ otherwise.} \end{aligned} \tag{43}$$

Here M_{BM} and V are the total baryonic mass and volume of the universe respectively.

From the forms of $G^0{}_0$ and $G^j{}_i$ it appears that the term $\frac{\beta}{\kappa}\phi^2$ represents pressure-less energy density i.e. $\phi^2 \propto a^{-3} \propto \frac{1}{e}$. Putting this in (41) we get after integration

$$h = -\gamma\phi^{\frac{8}{3}} + \frac{\lambda}{2\kappa} \tag{44}$$

where γ is a constant of dimension $(length)^{-\frac{4}{3}}$. Then from (42), we get

$$\begin{aligned} G^0{}_0 &= -3(\frac{\dot{a}}{a})^2 = -\kappa(\rho_{BM} + \rho_{DM} + \rho_{DR} + \rho_{VAC.}) \\ G^j{}_i &= -(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})\delta^j{}_i \\ &= \kappa(p_{BM} + p_{DM} + p_{DR} + p_{VAC.})\delta^j{}_i, \end{aligned} \tag{45}$$

where

$$p_{BM} = p_{DM} = 0 \tag{46}$$

$$\rho_{DM} = \frac{\beta}{\kappa}\phi^2 \tag{47}$$

$$\rho_{DR} = \frac{3\gamma}{2}\phi^{\frac{8}{3}}, \ \ p_{DR} = \frac{1}{3}\rho_R \tag{48}$$

$$\rho_{VAC.} = -p_{VAC.} = \frac{\lambda}{4\kappa} = \Lambda \text{ (say).} \tag{49}$$

As the scalar field ϕ, at present scale, appears to be non-interacting with the spinor field Ψ, *vide equations* (B13′), (B21′) & (B22′), the quantities having subscripts $_{BM}$, $_{DM}$, $_{DR}$ and $_{VAC.}$ may be assigned to the baryonic matter, the dark matter, the dark radiation and the vacuum

energy respectively. If we add another Lagrangian density to (31) corresponding to the Electro-Magnetic field and modify D by $D + A$, where A is the $U(1)$ connection one form, and also consider massless spinors having $c_\psi = 0$, then on the r. h. s. of equations in (45), ρ_{DR} and p_{DR} would be replaced by ρ_R and p_R containing various radiation components, given by

$$\begin{aligned} \rho_R &= \rho_{DR} + \rho_\gamma + \rho_\nu, \\ p_R &= p_{DR} + p_\gamma + p_\nu \qquad (50) \\ \text{s. t. } p_R &= \frac{1}{3}\rho_R, \qquad (51) \end{aligned}$$

where the subscripts have their usual meanings. Then from (39) and (45), we get

$$f = -\frac{1}{\kappa\phi^2\rho}(\frac{8}{3}\rho_{DR} + \frac{4}{3}\rho_{DM}), \qquad (52)$$

where ρ may be written as

$$\begin{aligned} \rho &= (1 + \frac{\rho_{BM}}{\rho_{DM}})\rho_{DM} + (1 + \frac{\rho_\gamma}{\rho_{DR}} + \frac{\rho_\nu}{\rho_{DR}})\rho_{DR} \\ &\quad + \rho_{VAC.} \qquad (53) \end{aligned}$$

Here the dimensionless parameter $\frac{\rho_{BM}}{\rho_{DM}}$ is the baryonic matter-dark matter ratio of the universe and, as both ρ_{BM} & ρ_{DM} have the same power-law of evolution at large cosmic scale, it may be taken to be a constant of time. Similarly the parameters $\frac{\rho_\gamma}{\rho_{DR}}$ and $\frac{\rho_\nu}{\rho_{DR}}$ may also be taken to be constants of time. And then from (52), f may be expressed in the following form,

$$f = -\frac{A + B\phi^{\frac{2}{3}}}{C\phi^2 + D\phi^{\frac{8}{3}} + E}, \qquad (54)$$

where A, B, C, D and E are constants having proper dimensions. It may be checked that $f \propto \phi^{-2}$(approx.) in both matter and radiation dominated era of the universe but f is nearly a constant at a very late time when the energy density is dominated by the cosmological constant.

5. ISOTROPIC, HOMOGENEOUS COSMOLOGY AND CONSERVED AXIAL CURRENT

In previous section we see that neglect of torsion coupling constant g leads to the standard FRW cosmology. Therefore if we assume, without g being negligible, that the back ground geometry may be extrapolated from that of standard FRW geometry then taking clue from equation (B21), we may postulate, hence after we call it as FRW-postulate,

$$f^*(d\phi \wedge {}^*d\phi) - 2h + \frac{\beta}{\kappa}\phi^2 = constant. \qquad (55)$$

In this case equation (B21) reduces to

$$\overline{\Psi}\gamma_5\gamma\Psi = -\kappa d^*(\overline{\Psi}\gamma_5\gamma\Psi \wedge T). \qquad (56)$$

Now defining an axial vector current 3-form, given by

$$J^A_{Tor.} = \kappa^*(\overline{\Psi}\gamma_5\gamma\Psi \wedge T)T, \qquad (57)$$

and using (B10) we get

$$dJ^A_{Tor.} = 0 \qquad (58)$$

6. NOETHER AXIAL CURRENT IN GENERAL CASE

In general, with out assuming any particular background geometry, we may define another axial vector current 3-form, given by

$$J^A_{EM} \equiv \kappa(\overline{\Psi}\gamma_5\gamma\Psi \wedge F) \qquad (59)$$

where $F = dA$ is the electro-magnetic or any $U(1)$ field strength. Then using equation (B21) we see that the current J^A_{EM} is conserved. Let us try to find out the symmetry involved, if any, with this conserved current.

Here we consider only the globally invariant part of the spinor Lagrangian $\mathcal{L}_D$ from (31) in curved space, given by

$$\begin{aligned} \overline{\mathcal{L}}_D &= \frac{i}{2}\{\overline{\Psi}{}^*\gamma \wedge d\Psi + \overline{d\Psi} \wedge {}^*\gamma\Psi\} - \frac{g}{4}\overline{\Psi}\gamma_5\gamma\Psi \wedge T \\ &\quad + c_\Psi\sqrt{{}^*dT}\overline{\psi}\Psi\eta \qquad (60) \end{aligned}$$

It can be verified that under global $SL(2, C)$ transformation on the spinor field and gamma matrices, given by,

$$\begin{aligned} \psi \to \Psi' &= S\Psi, \ \overline{\psi} \to \overline{\Psi'} = \overline{\Psi}S^{-1} \\ \text{and } \gamma \to \gamma' &= S\gamma S^{-1}, \qquad (61) \end{aligned}$$

where $S = \exp(\frac{i}{4}\theta_{ab}\sigma^{ab})$, $\overline{\mathcal{L}}_D$ is an invariant when θ_{ab} are constant numbers. Now imposing local invariance in the tangent space where $S = \exp(\frac{i}{4}\theta_{ab}\sigma^{ab}) = \exp(\frac{i}{4}\theta_{\mu\nu}\sigma^{\mu\nu})$, θ_{ab} are arbitrary infinetesimal tensor-indexed variables, we see that

$$\begin{aligned} 0 &= \delta\overline{\mathcal{L}}_D \\ &= -\frac{1}{8}\overline{\Psi}\{{}^*\gamma \wedge d(\theta_{\mu\nu}\sigma^{\mu\nu}) - d(\theta_{\mu\nu}\sigma^{\mu\nu}) \wedge {}^*\gamma\}\Psi \\ \Rightarrow \quad 0 &= d\theta \wedge (\overline{\Psi}\gamma_5\gamma\Psi) \\ &= d\{\theta \wedge (\overline{\Psi}\gamma_5\gamma\Psi)\} - \theta \wedge d(\overline{\Psi}\gamma_5\gamma\Psi). \qquad (62) \end{aligned}$$

Then arbitrariness of the 2-form θ implies $\overline{\Psi}\gamma_5\gamma\Psi$ to be a closed 1-form. From (B21$'$) we see that this 1-form is also exact. Being not a 3-form $\overline{\Psi}\gamma_5\gamma\Psi$ is not a physical conserved current. But we can introduce an infinitesimal local gauge transformation mediated by the background $U(1)$ field F, given by

$$\theta = \kappa\epsilon F \qquad (63)$$

where ϵ is an arbitrary infinitesimal scalar field. With this assumption, equation (62) reduces to

$$\begin{aligned} 0 &= d\{\kappa\epsilon F \wedge (\overline{\Psi}\gamma_5\gamma\Psi)\} \\ &\quad - \kappa\epsilon d(F \wedge \overline{\Psi}\gamma_5\gamma\Psi) \qquad (64) \end{aligned}$$

and then arbitrariness of ϵ implies J^A_{EM}, a physical conserved current 3-form, to be a Noether current.

We know the following theorem, [32, 33]:
Let $(M, g_{\mu\nu})$ be a spacetime associated with a gravity-matter system. Let, furthermore, $\Im$ be an initial hypersurface within an appropriate initial value problem. Then there exists a non-trivial Killing vector field ξ^μ and a gauge potential A^*_μ so that the Lie derivative $\pounds_\xi A^*_\mu = 0$ in a neighbourhood $\mathcal{O}$ of $\Im$, if and only if there exists a non-trivial initial data set $[\xi^\mu]$, satisfying $\bar{\nabla}_\mu \bar{\nabla}_\nu \xi_\alpha = R_{\nu\alpha\mu\beta}\xi^\beta$, so that $[\pounds_\xi g_{\mu\nu}]$ and $[\pounds_\xi A_\mu]$ vanish identically on Σ.
Owing to this existence theorem of the Killing vector field ξ there exists, [34, 35], a Noether current 3-form $\mathbf{J}$ and surface density 2-form $\mathbf{Q}$, given by

$$\mathbf{J} = d\mathbf{Q} \tag{65}$$
$$\mathbf{Q} = {}^*d\xi. \tag{66}$$

Then we can intoduce an expression for black hole entropy, [34, 35], given by

$$S_{Noether} \equiv \beta N = \beta \int_C \mathbf{J} = \beta \int_{\partial C} \mathbf{Q}, \tag{67}$$

here $\beta = \frac{2\pi}{\kappa_0}$, κ_0 is the surface gravity of the black hole horizion and C is an asymptotically flat hypersurface with "interior boundary"' ∂C. Now introducing the local $SL(2, C)$ gauge transformation on the spinor field and gamma matrices, mediated by the surface charge two form $d\xi$, where θ is given by

$$\theta = \epsilon d\xi. \tag{68}$$

Then the equation (62) reduces to

$$\begin{aligned} 0 &= d\{\epsilon d\xi \wedge (\overline{\Psi}\gamma_5\gamma\Psi)\} \\ &\quad -\epsilon d(d\xi \wedge \overline{\Psi}\gamma_5\gamma\Psi) \end{aligned} \tag{69}$$

and arbitrariness of ϵ establishes

$$J^A_{KV} \equiv d\xi \wedge \overline{\Psi}\gamma_5\gamma\Psi, \tag{70}$$

a physical current 3-form, to be a Noether current.

7. DISCUSSION

Here we see that if we introduce a scalar field ϕ to cause the de Sitter connection to have the proper dimension of a gauge field and also link this scalar field with the dimension of a Dirac field then we find that the Euler-Lagrange equations of both the fields to be mutually non-interacting. But they are indirectly connected to each other when we consider Euler-Lagrange equations of other geometric fields such as torsion and tetrad. Variation of the $SO(3,1)$ spin connection as an entity independent of the tetrads we get the Newton's constant as inversely proportional to the topological Nieh-Yan density and then the mass of the spinor field has been shown to be linked to the Newton's constant. Then using symmetries of the Einstein's tensor we get covariant conservation of angular momentum of the Dirac field in the particular class of geometry in U_4 as a generalization of the same in the Minkwoski space M_4. Neglecting the spin-torsion interaction term and considering FRW cosmology we are able to derive standard cosmology with standard energy density together with dark matter, dark radiation and cosmological constant.

Here we see that variation of torsion in the action gives us the axial vector 1-form $j_5 = \overline{\Psi}\gamma_5\gamma\Psi$ to be an exact form (B21$'$). If we consider FRW geometry to be in the background then the FRW postulate [Eqn.(55)][40] makes it possible to define an axial vector current 3-form $J^A_{Tor.}$ as a product of torsion and matter current j_5. $J^A_{Tor.}$ is conserved and as well as gauge invariant.

In manifolds having arbitrary background geometry the exterior product of j_5 and an $U(1)$ field strength F gives us a gauge invariant conserved current J^A_{EM}. Also, following the existance of a Killing vector field ξ, there exists another conserved current J^A_{KV}. Unlike conserved electric charge in standard model, where the vector current is given by $J \equiv d^*F$, conserved charge in the current $J^A_{EM} \equiv \kappa j_5 \wedge F$ is associated with the magnetic part of F. In a similar manner the current J^A_{KV} is associated with the magnetic part of $d\xi$ where as Noetherian entropy is connected to its electric part, [34–36]. In this sense, the conserved charges of J^A_{EM} and J^A_{KV} are connected to monopole charges.

Appendix A

From the definition of Lagrangian $\mathcal{L}_G$ in (25) we see that the $SO(3,1)$ connection $\bar{\omega}^{ab}$ is independent of the frame field e^a. It is metric dependent only on the on-shell, i.e. when we consider field equations corresponding to the variation of the Lagrange multipliers b_a. Hence following [38], we can independently vary e^a, $\overset{\omega}{\nabla} e^a$, dT, $\bar{R}^{ab}$, ϕ, $d\phi$, b^a and c^a and find

$$\begin{aligned} \delta\mathcal{L}_G &= \delta e^a \wedge \frac{\partial\mathcal{L}_G}{\partial e^a} + \delta \overset{\omega}{\nabla} e^a \wedge \frac{\partial\mathcal{L}_G}{\partial \overset{\omega}{\nabla} e^a} + \delta dT \frac{\partial\mathcal{L}_G}{\partial dT} \\ &\quad + \delta T \wedge \frac{\partial\mathcal{L}_G}{\partial T} + \delta\bar{R}^{ab} \wedge \frac{\partial\mathcal{L}_G}{\partial\bar{R}^{ab}} + \delta\phi\frac{\partial\mathcal{L}_G}{\partial\phi} \\ &\quad + \delta d\phi \wedge \frac{\partial\mathcal{L}_G}{\partial d\phi} + \delta b^a \wedge \frac{\partial\mathcal{L}_G}{\partial b^a} + \delta c^a \wedge \frac{\partial\mathcal{L}_G}{\partial c^a} \\ &= \delta e^a \wedge (\frac{\partial\mathcal{L}_G}{\partial e^a} + \overset{\omega}{\nabla} \frac{\partial\mathcal{L}_G}{\partial \overset{\omega}{\nabla} e^a}) + \delta T \wedge (d\frac{\partial\mathcal{L}_G}{\partial dT} \\ &\quad + \frac{\partial\mathcal{L}_G}{\partial T}) + \delta\bar{\omega}^{ab} \wedge \bar{\nabla}\frac{\partial\mathcal{L}_G}{\partial\bar{R}^{ab}} + \delta\omega^{ab} \wedge (\frac{\partial\mathcal{L}_G}{\partial \overset{\omega}{\nabla} e^a} \wedge e_b \\ &\quad + \frac{\partial\mathcal{L}_G}{\partial\omega^{ab}}) + \delta\phi(\frac{\partial\mathcal{L}_G}{\partial\phi} - d\frac{\partial\mathcal{L}_G}{\partial d\phi}) + \delta b^a \wedge \frac{\partial\mathcal{L}_G}{\partial b^a} \\ &\quad + \delta c^a \wedge \frac{\partial\mathcal{L}_G}{\partial c^a} + d(\delta e^a \wedge \frac{\partial\mathcal{L}_G}{\partial\bar{\nabla}e^a} + \delta T\frac{\partial\mathcal{L}_G}{\partial dT} \end{aligned} \tag{A1}$$

$$+\delta\bar{\omega}^{ab}\wedge\frac{\partial\mathcal{L}_G}{\partial\bar{R}^{ab}}+\delta\phi\frac{\partial\mathcal{L}_G}{\partial d\phi}) \qquad (A2)$$

Using the form of the Lagrangian $\mathcal{L}_G$, given in (25), we get

$$\begin{aligned}\frac{\partial\mathcal{L}_G}{\partial e^a} &= -\frac{1}{6}{}^*N(2\mathbf{R}_a-\mathcal{R}\eta_a)-{}^*[b_b\wedge(\overset{\omega}{\nabla}e^b-T^b]^2\eta_a\\ &\quad -f(\phi)[-2\partial_a\phi\partial^b\phi\eta_b+\partial_b\phi\partial^b\phi\eta_a]\\ &\quad -h(\phi)\eta_a \qquad (A3)\\ \frac{\partial\mathcal{L}_G}{\partial(\bar{\nabla}e^a)} &= 2^*[b_b\wedge(\overset{\omega}{\nabla}e^b-T^b)]b_a \qquad (A4)\\ \frac{\partial\mathcal{L}_G}{\partial(dT)} &= \mathcal{R}-\beta\phi^2 \qquad (A5)\\ \frac{\partial\mathcal{L}_G}{\partial\bar{R}^{ab}} &= -\frac{1}{24}{}^*N\epsilon_{abcd}e^c\wedge e^d=-\frac{1}{12}{}^*N\eta_{ab} \qquad (A6)\\ \frac{\partial\mathcal{L}_G}{\partial\phi} &= -\frac{1}{3}\beta\phi N-f'(\phi)d\phi\wedge{}^*d\phi-h'(\phi)\eta \qquad (A7)\\ \frac{\partial\mathcal{L}_G}{\partial d\phi} &= -2f^*d\phi \qquad (A8)\\ \frac{\partial\mathcal{L}_G}{\partial b^a} &= 2^*[b_a\wedge(\overset{\omega}{\nabla}e^a-T^a)][(\overset{\omega}{\nabla}e^a-T^a)] \qquad (A9)\\ \frac{\partial\mathcal{L}_G}{\partial c^a} &= 2^*[c_a\wedge(\omega^{ab}-\bar{\omega}^{ab}-T^{ab})][(\omega^{ab}\\ &\quad -\bar{\omega}^{ab}-T^{ab})] \qquad (A10)\end{aligned}$$

Where

$$\begin{aligned}\mathbf{R}_a:=\frac{1}{2}\frac{\partial(\mathcal{R}\eta)}{\partial e^a} &= \frac{1}{4}\epsilon_{abcd}\bar{R}^{bc}\wedge e^d\\ \text{and }\ \partial_a &= e_a{}^\mu\frac{\partial}{\partial x^\mu}. \qquad (A11)\end{aligned}$$

From above, Euler-Lagrange equations for b_a and c_a imply $\overset{\omega}{\nabla}e^a=T^a$ as torsion and the connection $\bar{\omega}^{ab}=\omega^{ab}-T^{ab}$ as torsion free, i.e.

$$\bar{\nabla}e_a = 0. \qquad (A12)$$

Using this result in (A3) and (A4) we get

$$\begin{aligned}\frac{\partial\mathcal{L}_G}{\partial e^a} &= -\frac{1}{6}{}^*N(2\mathbf{R}_a-\mathcal{R}\eta_a)-f(\phi)[-2\partial_a\phi\partial^b\phi\eta_b\\ &\quad +\partial_b\phi\partial^b\phi\eta_a]-h(\phi)\eta_a \qquad (A13)\\ \frac{\partial\mathcal{L}_G}{\partial(\bar{\nabla}e^a)} &= 0 \qquad (A14)\end{aligned}$$

Appendix B

From the definition of the Lagrangian $\mathcal{L}_D$ in (31) we see that in the covarint derivative $D\psi$ the connection 1-form Γ is a function of the frame fields and their derivatives. So we can't treat $D\psi$ and e^a as independent fields. Keeping this in mind we consider the variation of $\mathcal{L}_D$ with respect to its independent fields ψ, $\overline{\psi}$, $d\psi$, $\overline{d\psi}$, T, dT, ϕ, e^a and de^a and can write

$$\begin{aligned}\delta\mathcal{L}_D &= \frac{\partial\mathcal{L}_D}{\partial\psi}\delta\psi+\frac{\partial\mathcal{L}_D}{\partial(D\psi)}\wedge\delta(D\psi)+\delta\overline{\psi}\frac{\partial\mathcal{L}_D}{\partial\overline{\psi}}\\ &\quad +\delta\overline{D\psi}\wedge\frac{\partial\mathcal{L}_D}{\partial\overline{D\psi}}+\delta T\wedge\frac{\partial\mathcal{L}_D}{\partial T}+\delta(dT)\frac{\partial\mathcal{L}_D}{\partial(dT)}\\ &\quad +\delta\phi\frac{\partial\mathcal{L}_D}{\partial\phi}+\delta e^a\wedge\frac{\partial\mathcal{L}_D}{\partial e^a} \qquad (B1)\end{aligned}$$

Here some variations e^a and de^a are contained in $\delta\Gamma$ of δD. Now $\delta(D\psi)=D\delta\psi+\delta\Gamma\psi$, since d and δ commute, reduces (B1) to

$$\begin{aligned}\delta\mathcal{L}_D &= \phi^2[\{i\overline{D\psi}\wedge{}^*\gamma+id\ln\phi\wedge\overline{\psi}^*\gamma-\frac{g}{4}\overline{\psi}\gamma_5\gamma\wedge T\\ &\quad +c_\psi\sqrt{{}^*dT}\overline{\psi}\eta\}\delta\psi+\delta\overline{\psi}\{i^*\gamma\wedge D\psi\\ &\quad +i^*\gamma\psi\wedge d\ln\phi-\frac{g}{4}\gamma_5\gamma\wedge T\psi+c_\psi\sqrt{{}^*dT}\psi\eta\}\\ &\quad +\frac{i}{2}\{\overline{\delta\Gamma\psi}\wedge{}^*\gamma\psi+\overline{\psi}^*\gamma\wedge\delta\Gamma\psi\}\\ &\quad +\frac{i}{2}\{\overline{\psi}\delta({}^*\gamma)\wedge D\psi+\overline{D\psi}\wedge\delta({}^*\gamma)\psi\}\\ &\quad +\delta T\wedge\{\frac{g}{4}\overline{\psi}\gamma_5\gamma\psi-\frac{1}{2\phi^2}c_\psi d(\frac{\phi^2}{\sqrt{{}^*dT}}\overline{\psi}\psi)\}\\ &\quad +\delta e^a\wedge\{-\frac{g}{4}\overline{\psi}\gamma_5\gamma_a\psi T+\frac{1}{2}c_\psi\sqrt{{}^*dT}\overline{\psi}\psi\eta_a\}]\\ &\quad +2\phi[\frac{i}{2}\{\overline{\psi}^*\gamma\wedge D\psi+\overline{D\psi}\wedge{}^*\gamma\psi\}\\ &\quad -\frac{g}{4}\overline{\psi}\gamma_5\gamma\psi\wedge T+c_\psi\sqrt{{}^*dT}\overline{\psi}\psi\eta]\delta\phi\\ &\quad +\ \text{surface terms(S. T.)}. \qquad (B2)\end{aligned}$$

Third term of this equation gives

$$\begin{aligned}&\frac{i}{2}[\overline{\delta\Gamma\psi}\wedge{}^*\gamma\psi+\overline{\psi}^*\gamma\wedge\delta\Gamma\psi]\phi^2\\ &=\frac{i}{2}\overline{\psi}[-\delta\Gamma\wedge{}^*\gamma+{}^*\gamma\wedge\delta\Gamma]\psi\phi^2\\ &=-\frac{1}{8}\overline{\psi}[\sigma_{cb}\delta(e^{c\mu}e^b{}_{\mu:\nu})\gamma^\nu+\gamma^\nu\sigma_{cb}\delta(e^{c\mu}e^b{}_{\mu:\nu})]\psi\phi^2\eta\\ &=\frac{1}{8}\delta e^a\wedge[\overline{\psi}(\sigma_{cb}e_a{}^\mu e^b{}_{\mu:\nu}\gamma^\nu+\gamma^\nu\sigma_{cb}e_a{}^\mu e^b{}_{\mu:\nu})\psi\phi^2\eta^c\\ &\quad +D_\nu^{\{\}}\{\overline{\psi}(\sigma_{ca}\gamma^\nu+\gamma^\nu\sigma_{ca})\psi\phi^2\eta^c\}]+\ \text{(S. T.)} \qquad (B3)\end{aligned}$$

It is to be noted here that, using the properties of γ matrices, the only surviving terms of $\frac{i}{2}[\overline{\Gamma\psi}\wedge{}^*\gamma\psi+\overline{\psi}^*\gamma\wedge\Gamma\psi]=-\frac{i}{2}\overline{\psi}(\gamma^\mu\gamma_{\mu:\nu}\gamma^\nu+\gamma^\nu\gamma^\mu\gamma_{\mu:\nu})\psi\eta$ are those for which γ^μ, γ^ν and $\gamma_{\mu:\nu}$ are anti-symmetrized. This implies that, in the variational calculation, the Christoffel part of the Riemannian covariant derivative remains insensitive.

Fourth term of (B2) gives

$$\begin{aligned}&\frac{i}{2}\{\overline{\psi}\delta({}^*\gamma)\wedge D\psi+\overline{D\psi}\wedge\delta({}^*\gamma)\psi\}\phi^2\\ &=\frac{i}{2}\delta e^a\wedge\{\overline{\psi}\gamma^b\eta_{ba}\wedge D\psi-\overline{D\psi}\gamma^b\psi\wedge\eta_{ba}\}\phi^2 \qquad (B4)\end{aligned}$$

Hence Euler-Lagrange equations corresponding to the extremum of $\mathcal{L}_{tot.}$ from the independent variations of e^a, T, ϕ, ω^{ab} and $\bar{\omega}^{ab}$, using (A2), (A5) and (A6), give us

$$\begin{aligned}
&\frac{1}{6}{}^*N(2\mathbf{R}_a - \mathcal{R}\eta_a)\\
&+f(\phi)[-2\partial_a\phi\partial^b\phi\eta_b + \partial_b\phi\partial^b\phi\eta_a] + h(\phi)\eta_a\\
&-\frac{1}{8}[\overline{\psi}(\sigma_{cb}e_a{}^\mu e^b{}_{\mu:\nu}\gamma^\nu + \gamma^\nu\sigma_{cb}e_a{}^\mu e^b{}_{\mu:\nu})\psi\phi^2\eta^c\\
&+D_\nu^{\{\}}\{\overline{\psi}(\sigma_{ca}\gamma^\nu + \gamma^\nu\sigma_{ca})\psi\phi^2\eta^c\}]\\
&+[\frac{g}{4}\overline{\psi}\gamma_5\gamma_a\psi\wedge T - \frac{1}{2}c_\psi\sqrt{{}^*dT}\overline{\psi}\psi\eta_a]\phi^2\\
&-\frac{i}{2}\{\overline{\psi}\gamma^b\eta_{ba}\wedge D\psi - \overline{D\psi}\gamma^b\psi\wedge\eta_{ba}\}\phi^2 = 0
\end{aligned} \quad \text{(B5)}$$

$$d(\mathcal{R} - \beta\phi^2 - c_\psi\frac{\phi^2}{2\sqrt{{}^*dT}}\overline{\psi}\psi) = -\frac{g}{4}\phi^2\overline{\psi}\gamma_5\gamma\psi \quad \text{(B6)}$$

$$\begin{aligned}
&-2\beta\phi N + f'(\phi)d\phi\wedge{}^*d\phi - h'(\phi)\eta + 2fd^*d\phi\\
&= -2\phi[\frac{i}{2}\{\overline{\psi}^*\gamma\wedge D\psi + \overline{D\psi}\wedge{}^*\gamma\psi\}\\
&\quad -\frac{g}{4}\overline{\psi}\gamma_5\gamma\psi\wedge T + c_\psi\sqrt{{}^*dT}\overline{\psi}\psi\eta]
\end{aligned} \quad \text{(B7)}$$

$$\bar{\nabla}({}^*N\eta_{ab}) = 0 \quad \text{(B8)}$$

Using (A12) in (B8), we get

$$d^*N = 0 \quad \text{(B9)}$$

From this equation we can write

$${}^*N = \frac{6}{\kappa} \quad \text{(B10)}$$

where κ is an integration constant having $(length)^2$ dimension. From (B2) we can write the Euler-Lagrange equations for the fields ψ and $\overline{\psi}$ as

$$\begin{aligned}
&i^*\gamma\wedge D\psi + i^*\gamma\psi\wedge d\ln\phi\\
&-\frac{g}{4}\gamma_5\gamma\wedge T\psi + m_\psi\psi\eta = 0,\\
&i\overline{D\psi}\wedge{}^*\gamma + id\ln\phi\wedge\overline{\psi}^*\gamma\\
&-\frac{g}{4}\overline{\psi}\gamma_5\gamma\wedge T + m_\psi\overline{\psi}\eta = 0,
\end{aligned} \quad \text{(B11)}$$

provided we define, using (B10), the mass of the field ψ as

$$m_\psi = c_\psi\sqrt{{}^*dT} = \frac{c_\psi}{\sqrt{\kappa}} \quad \text{(B12)}$$

If we define $\Psi = \phi\psi$ as the Dirac field having the proper dimension and conformal weight and $m_\Psi = m_\psi$, the equations in (B11) reduce to their standard form in the particular class of geometry in U_4 space[39]

$$\begin{aligned}
i^*\gamma\wedge D\Psi - \frac{g}{4}\gamma_5\gamma\wedge T\Psi + m_\Psi\Psi\eta &= 0,\\
i\overline{D\Psi}\wedge{}^*\gamma - \frac{g}{4}\overline{\Psi}\gamma_5\gamma\wedge T + m_\Psi\overline{\Psi}\eta &= 0.
\end{aligned} \quad \text{(B13)}$$

Now using (B10), (B12) and (B13), the field equation (B5) reduces to

$$\begin{aligned}
&\frac{1}{\kappa}(2\mathbf{R}_a - \mathcal{R}\eta_a) + f(\phi)[-2\partial_a\phi\partial^b\phi\eta_b + \partial_b\phi\partial^b\phi\eta_a]\\
&+h(\phi)\eta_a - \frac{1}{8}\overline{\nabla}_\nu\{\overline{\Psi}(\sigma_{ca}\gamma^\nu + \gamma^\nu\sigma_{ca})\Psi\eta^c\}\\
&+[\frac{g}{4}\overline{\Psi}\gamma_5\gamma_a\Psi\wedge T - \frac{1}{2}m_\Psi\overline{\Psi}\Psi\eta_a]\\
&-\frac{i}{2}\{\overline{\Psi}\gamma^b\eta_{ba}\wedge D\Psi - \overline{D\Psi}\gamma^b\Psi\wedge\eta_{ba}\} = 0
\end{aligned} \quad \text{(B14)}$$

where $\overline{\nabla}_\nu$ represents torsion-free covariant differentiation w. r. t. both external and internal indices. Now by exterior multiplication with e^a from left, this equation yields

$$\begin{aligned}
&\frac{2}{\kappa}\mathcal{R}\eta + \frac{3i}{2}\{\overline{\Psi}^*\gamma\wedge D\Psi + \overline{D\Psi}\wedge{}^*\gamma\Psi\} - \frac{g}{4}\overline{\Psi}\gamma_5\gamma\Psi\wedge T\\
&+2m_\Psi\overline{\Psi}\Psi\eta - 2fd\phi\wedge{}^*d\phi - 4h\eta = 0
\end{aligned} \quad \text{(B15)}$$

and after using Dirac equations (B13), we get

$$\begin{aligned}
\mathcal{R}\eta = \kappa[&\frac{1}{2}m_\Psi\overline{\Psi}\Psi\eta - \frac{g}{4}\overline{\Psi}\gamma_5\gamma\Psi\wedge T\\
&+fd\phi\wedge{}^*d\phi + 2h\eta]
\end{aligned} \quad \text{(B16)}$$

Again, taking exterior multiplication of (B14) by e^b, we get

$$\begin{aligned}
&\frac{1}{\kappa}[\mathcal{R}\delta^b{}_a + 2G^b{}_a]\eta + \frac{i}{2}\{\overline{\Psi}^*\gamma\wedge D\Psi + \overline{D\Psi}\wedge{}^*\gamma\Psi\}\delta^b{}_a\\
&+\frac{i}{2}\{\overline{\Psi}\gamma^b D_a\Psi - \overline{D_a\Psi}\gamma^b\Psi\}\eta - \frac{g}{4}\overline{\Psi}\gamma_5\gamma_a\Psi^*T^b\eta\\
&+\frac{1}{2}m_\Psi\overline{\Psi}\Psi\delta^b{}_a\eta + \frac{1}{8}\overline{\nabla}_\nu\{\overline{\Psi}(\sigma^b{}_a\gamma^\nu + \gamma^\nu\sigma^b{}_a)\Psi\}\eta\\
&+2f\partial_a\phi\partial^b\phi\eta - fd\phi\wedge{}^*d\phi\delta^b{}_a - h\eta\delta^b{}_a = 0
\end{aligned} \quad \text{(B17)}$$

Using Dirac equations (B13) and equation (B16), this equation reduces to

$$\begin{aligned}
G^b{}_a\eta = -\kappa[&\frac{i}{4}\{\overline{\Psi}\gamma^b D_a\Psi - \overline{D_a\Psi}\gamma^b\Psi\}\eta\\
&-\frac{g}{8}\overline{\Psi}\gamma_5\gamma_a\Psi^*T^b\eta\\
&+\frac{1}{16}\overline{\nabla}_\nu\{\overline{\Psi}(\sigma^b{}_a\gamma^\nu + \gamma^\nu\sigma^b{}_a)\Psi\}\eta\\
&+f\partial_a\phi\partial^b\phi\eta + \frac{1}{2}(h)\eta\delta^b{}_a],
\end{aligned} \quad \text{(B18)}$$

here ${}^*T_a$ is the flat space tensorial component of the one form ${}^*T$. Now using symmetries of the Einstein's tensor G^{ab}, we can break this equation to a symmetric part and an antisymmetric part, and can write

$$\begin{aligned}
G^b{}_a\eta = -\kappa[&\frac{i}{8}\{\overline{\Psi}(\gamma^b D_a + \gamma_a D^b)\Psi - (\overline{D_a\Psi}\gamma^b +\\
&\overline{D^b\Psi}\gamma_a)\Psi\}\eta - \frac{g}{16}\overline{\Psi}\gamma_5(\gamma_a{}^*T^b + \gamma^{b*}T_a)\Psi\eta\\
&+f\partial_a\phi\partial^b\phi\eta + \frac{1}{2}(h)\eta\delta^b{}_a],
\end{aligned} \quad \text{(B19)}$$

$$0 = [\frac{1}{2}\overline{\nabla}_\nu\overline{\Psi}\{\frac{\sigma^b{}_a}{2},\gamma^\nu\}\Psi + \frac{i}{2}\{\overline{\Psi}(\gamma^b D_a - \gamma_a D^b)\Psi - (\overline{D_a\Psi}\gamma^b - \overline{D^b\Psi}\gamma_a)\Psi\} - \frac{g}{4}\overline{\Psi}\gamma_5(\gamma_a{}^*T^b - \gamma^{b*}T_a)\Psi]\eta. \quad \text{(B20)}$$

Again using (B10), (B11) and (B12), equations (B6) and (B7) reduce to

$$d(\mathcal{R} - \beta\phi^2 - \frac{1}{2}\kappa m_\Psi\overline{\Psi}\Psi) = -\frac{g}{4}\overline{\Psi}\gamma_5\gamma\Psi$$

$$\text{or,}\quad \kappa d[\frac{g}{4}{}^*(\overline{\Psi}\gamma_5\gamma\Psi\wedge T) - f^*(d\phi\wedge{}^*d\phi) + 2h - \frac{\beta}{\kappa}\phi^2] = -\frac{g}{4}\overline{\Psi}\gamma_5\gamma\Psi, \quad \text{(B21)}$$

$$\frac{2}{\kappa}\beta\phi + f'(\phi)d\phi\wedge{}^*d\phi - h'(\phi)\eta + 2fd^*d\phi = -2\phi[\frac{i}{2}\{\overline{\psi}^*\gamma\wedge D\psi + \overline{D\psi}\wedge{}^*\gamma\psi\} - \frac{g}{4}\overline{\psi}\gamma_5\gamma\psi\wedge T + m_\psi\overline{\psi}\psi\eta] = 0. \quad \text{(B22)}$$

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